Minimal Positive Polynomials*

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***Summary*—A proof is given of a purely mathematical theorem on the polynomial of lowest degree with positive coefficients having a prescribed root of unity as a multiple root.**

H. J. Riblet has conjectured the theorem below. In the preceding paper,[1] he applies his theorem to optimum impedance transformer design.

Theorem

THE polynomial $(x^r+1)^n$ is the unique monic polynomial of lowest degree with non-negative coefficients that has $e^{i\pi/r}$ as an n-fold root.

The degree of the minimal positive polynomial is at most nr. Let the polynomial be

$$f(x) = a_{nr}x^{nr} + a_{nr-1}x^{nr-1} + \cdots + a_0.$$

If $n=1$, the imaginary part of $f(e^{i\pi/r})$, of necessity zero, may be written as

$$a_{r-1} \sin (r - 1)\pi/r + \cdots + a_1 \sin \pi/r.$$

Therefore, $a_k=0$, $0<k<r$, for in this range $0<\sin k\pi/r$. The real part, also zero, now reduces to $-a_r+a_0$. Hence, $f(x)=a_r(x^r+1)$.

For larger n, note that if $e^{i\pi/r}$ is an n-fold root of $f(x)$ it is also a root of the first $n-1$ derivatives of $f(x)$. It is then a root of

$$F(x) = b_0 f(x) + b_1 x f'(x) + \cdots + b_{n-1}x^{n-1}f^{(n-1)}(x)$$

for any choice of the constants b_i. $F(x)$ may be rewritten as

$$F(x) = a_{nr}g(nr)x^{nr} + a_{nr-1}g(nr - 1)x^{nr-1} + \cdots + a_0 g(0) \quad (1)$$

where

$$g(k) = b_0 + b_1 k + b_2 k(k - 1) + \cdots + b_{n-1}k(k - 1) \cdots (k - n + 2). \quad (2)$$

Although, in so far as $g(k)$ appears in (1), k is restricted to certain integral values. Eq. (2) defines $g(k)$ as a polynomial in k of degree $n-1$.

In (1) replace x by $e^{i\pi/r}$ and compute the imaginary part of $F(x)$ to obtain

$$a_{nr}g(nr) \sin nr\pi/r + a_{nr-1}g(nr - 1) \sin (nr - 1)\pi/r + \cdots + a_1 g(1) \sin \pi/r = 0. \quad (3)$$

The constants b_i in (2) may be chosen so that $g(k)$ has the same sign as $\sin k\pi/r$, $0<k<nr$. Since a_k is not negative, this will imply that $a_k g(k) \sin k\pi/r$ also is not negative and hence, by (3), zero.

The proper behavior of $g(k)$ is readily obtained by requiring that $g(k)=0$, $k=r, 2r, \cdots, (n-1)r$. This leads to a system of $n-1$ linear homogeneous equations in the n quantities $b_0, b_1, \cdots, b_{n-1}$. The system has nontrivial solutions and one may be selected for which $g(1)$ is positive. Since $g(k)$ is of degree $n-1$, it has no other roots and $g(k) \sin k\pi/r \geq 0$. Therefore, $a_k=0$ if k is not divisible by r. Thus, $f(x)$ is a polynomial in x^r: $f(x)=h(x^r)$. It has the n-fold root $e^{i\pi/r}$ and $h(x)$ accordingly has the n-fold root $e^{i\pi}$. This shows that $h(x)=a_{nr}(x+1)^n$ and $f(x)=a_{nr}(x^r+1)^n$.

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[1] H. J. Riblet, "A general theorem on an optimum stepped impedance transformer," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, this issue, pp. 169–170.